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Thermal convection in a vertical circular cylinder

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A theoretical investigation is made of the onset of buoyancy-driven convection in a circular cylinder. Amplitude equations are derived for the weakly nonlinear evolution of critical disturbances at moderate values of the radius-to-height ratio. It is shown that the initial form of the convective motion at Rayleigh numbers slightly above critical is not axisymmetric. Particular attention is paid to the neighbourhoods of points where two disturbances are simultaneously critical according to linear theory; the nonlinear evolution in such neighbourhoods is studied in detail.

1. Introduction

We study the onset of buoyancy-driven convection in a layer of Boussinesq fluid having the shape of a cylinder of uniform circular cross-section, with its axis in the vertical direction and with flat top and bottom. The parameter describing the geometry of the layer is the ratio of cylinder radius to cylinder height, henceforth called the aspect ratio and denoted by a. The layer is heated from below, and the other principal parameter is the Rayleigh number R.

Our main purpose is to investigate how the weakly nonlinear interactions that generate convection are modified as the aspect ratio changes. In this paper we focus attention on moderate aspect ratios, in the approximate interval $0{\cdot}5 < a < 2{\cdot}0$; different effects come into play at both very small and very large aspect ratios, and will be studied separately elsewhere. We take full account in our analysis of non-axisymmetric modes of convection, rather than concentrating on the axisymmetric mode as previous workers, such as Liang, Vidal & Acrivos (1969) and Charlson & Sani (1975), have tended to do. Our principal results are that, in the range of aspect ratios under consideration, the convective motion appearing at Rayleigh numbers just above critical is *not* axisymmetric; when $a < 1{\cdot}0$ approximately the convection has unit azimuthal wavenumber, while when $1{\cdot}0 < a < 2{\cdot}0$ there is a tendency for two modes to occur simultaneously, either independently or as a mixed-mode combination.

At first sight it might appear that this result is inconsistent with experimental observations (see e.g. the review by Koschmieder 1974), which generally allude to the appearance of circular rings, that is, axisymmetric solutions. However, most of the experimental results are for cylinders of very large aspect ratio, outside the range considered in this paper. On the other hand, Mitchell & Quinn (1965) saw non-axisymmetric modes in a cylinder of aspect ratio near one, which is consistent with our findings. It is also noteworthy that calculations by Charlson & Sani (1975) have shown that under certain circumstances an axisymmetric solution becomes unstable to a non-axisymmetric disturbance in a container of moderate aspect ratio.

The governing equations and boundary conditions are set out in § 2. In order to reduce drastically the level of algebraic and computational complexity we apply artificial sidewall boundary conditions that permit the linear stability problem to be solved by separation of variables. Although these boundary conditions are unphysical, it seems nevertheless likely that the qualitative conclusions will remain valid for other, more realistic boundary conditions.

In § 3 the critical Rayleigh number is determined as a function of the aspect ratio, and also of the mode of instability. We find that, as the aspect ratio increases, the azimuthal wavenumber of the critical mode changes from 1 to 2 to 0 to 3 in the range $0 < a < 2{\cdot}0$. This ordering is similar to that found by Charlson & Sani (1971) for the Bénard problem with different boundary conditions, and by Rosenblat, Davis & Homsy (1982) for the analogous Marangoni problem.

The nonlinear evolution problem is tackled by a method, outlined in § 4, in which field quantities are expanded in series of eigenfunctions of the linear stability problem. This approach was developed by Eckhaus (1965), and the criterion for truncation of the series used here follows a proposal by Rosenblat (1979).

Application of this method leads eventually to a nonlinear evolution equation of Landau type when a single mode is critical, and to a pair of coupled Landau equations at points where two modes are simultaneously critical. The nature and stability of the solutions to such equations are examined in §§ 5 and 6 respectively. A brief discussion of the results is presented in § 7.

2. Formulation

We consider a Boussinesq fluid with constant viscosity and thermal diffusivity, and whose density varies linearly with temperature. The fluid occupies a cylindrical container of uniform circular cross-section, with its axis parallel to the direction of gravity. We employ a dimensionless cylindrical polar coordinate system $\mathbf{r} = (r, \phi, z)$, with its origin at the centre of the lower boundary; we take the upper boundary to be at $z = 1$, and the lateral boundary at $r = a$, so that the aspect ratio is a.

For heating from below there is a (motionless) conduction solution to the governing equations (Navier–Stokes, continuity and energy, with the Boussinesq approximation) in which both temperature and density are linear functions of the vertical coordinate z, and independent of r and ϕ. Under the circumstances described this is the same solution as for a horizontally unbounded layer.

When disturbances are applied to the conduction state, the dimensionless governing equations for the disturbance field quantities are easily obtained and well known. They are

$$\nabla^2\mathbf{v} - \nabla p + R^{\frac{1}{2}}\theta\hat{\mathbf{z}} = Pr^{-1}\left\{\frac{\partial v}{\partial t} + (\mathbf{v}\,.\,\nabla)\,\mathbf{v}\right\}, \tag{2.1}$$

$$\nabla\,.\,\mathbf{v} = 0, \tag{2.2}$$

$$\nabla^2\theta + R^{\frac{1}{2}}w = \frac{\partial \theta}{\partial t} + (\mathbf{v}\,.\,\nabla)\,\theta, \tag{2.3}$$

where $\mathbf{v}, p, \theta$ are dimensionless disturbance velocity, pressure and temperature respectively, w is the z-component of velocity, $\mathbf{v} = (u, v, w)$ in cylindrical polars, $\hat{\mathbf{z}}$ is the unit vector in the z-direction, R is the Rayleigh number and Pr is the Prandtl number.

We assume that the upper and lower horizontal boundaries are isothermal, and that they are non-deformable free surfaces. This leads to the boundary conditions

$$\theta = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{on} \quad z = 0,\ 1, \quad 0 \leqslant r < a. \tag{2.4}$$

The lateral boundary is assumed to be perfectly insulating, and to be a nondeformable surface on which the tangential vorticity is zero. As a result we find

$$\frac{\partial \theta}{\partial r} = u = \frac{\partial}{\partial r}(rv) = \frac{\partial w}{\partial r} = 0 \quad \text{on} \quad r = a, \quad 0 < z < 1. \tag{2.5}$$

As mentioned in § 1, these artificial boundary conditions allow considerable simplification of the mathematical analysis, but seem unlikely to yield qualitatively misleading information regarding realistic physical behaviour.

3. Linear stability

The value of the Rayleigh number at which the conduction solution loses stability is determined from linearization of the system (2.1)–(2.3), together with the boundary conditions (2.4)–(2.5). It is easy to show that this linear problem is self-adjoint, which implies that the principle of exchange of stabilities holds. Hence we need only consider (2.1)–(2.3) with the right-hand sides set equal to zero, subject to (2.4)–(2.5).

The boundary-value problem is solved by standard methods. A solution can be obtained in which the variables are separated, and then some elementary calculations lead to the result that the eigenvalues of the problem are given by

$$R = \frac{[(n\pi)^2 + \lambda^2]^3}{\lambda^2}. \tag{3.1}$$

Here $n = 1, 2, 3, \ldots$ is associated with the number of zeros of the corresponding eigensolution in the z-direction, and is called the *vertical wavenumber*; λ is a real positive number determined by the condition

$$J'_m(\lambda a) = 0, \tag{3.2}$$

where $m = 0, 1, 2, \ldots$ is the *azimuthal wavenumber*, and J_m is the usual Bessel function of order m.

The critical Rayleigh number for the onset of convection is determined in the following way. Let s_{mj} $(j = 1, 2, \ldots)$ denote the jth positive root of the equation

$$J'_m(s) = 0 \quad (m = 0, 1, 2, \ldots), \tag{3.3}$$

and write

$$\lambda_{mj} = s_{mj}/a. \tag{3.4}$$

(We shall call $j = 1, 2, \ldots$ *the radial wavenumber*.) Then for each value of m, j and n the boundary-value problem admits an eigensolution, with the eigenvalue

$$R_{mjn} = R_{mjn}(a) = \frac{[(n\pi)^2 + \lambda_{mj}^2]^3}{\lambda_{mj}^2} \tag{3.5}$$

for $m = 0, 1, 2, \ldots$ and $j, n = 1, 2, 3, \ldots$. The critical Rayleigh number at a fixed aspect ratio a is then defined to be

$$R_c \doteq R_c(a) = \min_{m,j,n} R_{mjn}(a). \tag{3.6}$$

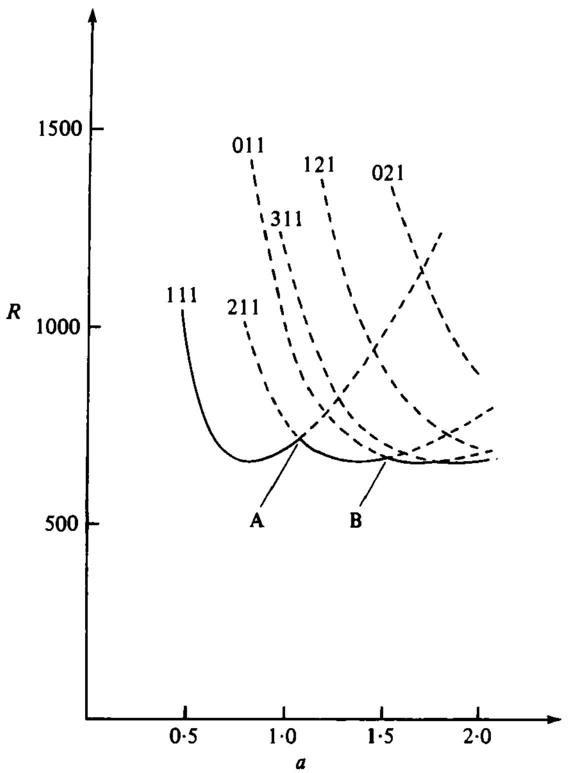

FIGURE 1. Rayleigh number for neutral stability as a function of aspect ratio for different disturbance modes.

The solid curve in figure 1 depicts the critical Rayleigh number as a function of aspect ratio in the range $0{\cdot}5 \leqslant a \leqslant 2{\cdot}0$. The dotted curves show the eigenvalues (3.5) for various indicated values of m, j, n. We see that for small aspect ratios, for $a < 1{\cdot}1$ approximately, the critical Rayleigh number corresponds to a mode with azimuthal wavenumber $m = 1$. Next, in the range $1{\cdot}1 < a < 1{\cdot}55$, the critical mode has azimuthal wavenumber $m = 2$; with further increase of aspect ratio $(1{\cdot}55 < a < 1{\cdot}8)$ the critical mode is axisymmetric, with $m = 0$. In the case of rigid sidewalls, Charlson & Sani (1971) showed that the ordering of the modes was principally determined by the ordering of the zeros of a Bessel function. In the case of insulating boundaries, their characteristic equation was identical with (3.3). This fact lends some justification to our use of artificial boundary conditions to exhibit qualitative behavioural features. Figure 1 is also very similar to the linear stability diagram computed by Rosenblat *et al.* (1982) for the analogous Marangoni problem.

For future reference we write down the eigenfunction corresponding to the eigenvalue (3.5) for each m, j, n. A single calculation gives the following expressions:

$$u_{mjn} = C_{mjn}(n\pi/\lambda_{mj}) \cos m\phi \, J'_m(\lambda_{mj} r) \cos n\pi z, \tag{3.7a}$$

$$v_{mjn} = -C_{mjn}(mn\pi/\lambda^2_{mj} r) \sin m\phi \, J_m(\lambda_{mj} r) \cos n\pi z, \tag{3.7b}$$

$$w_{mjn} = C_{mjn} \cos m\phi \, J_m(\lambda_{mj} r) \sin n\pi z, \tag{3.7c}$$

$$\theta_{mjn} = \cos m\phi \, J_m(\lambda_{mj} r) \sin n\pi z, \tag{3.7d}$$

with C_{mjn} given by

$$C_{mjn} = \frac{\lambda_{mj}}{[(n\pi)^2 + \lambda_{mj}^2]^{\frac{1}{2}}}. \tag{3.8}$$

As noted earlier, the linear stability problem is self-adjoint, so that (3.7) also gives the eigensolutions of the adjoint problem.

4. Nonlinear convection

We turn our attention now to the convective motions that arise in association with the loss of stability of the conduction solution. We investigate these motions at various values of the aspect ratio in the range $0{\cdot}5 < a < 2{\cdot}0$, and for slightly supercritical Rayleigh numbers. The nonlinear problem will be tackled with what is essentially a type of Galerkin approximation in which the eigenvalues of the linear stability problem are used as the basis for an expansion procedure. This method resembles an approach suggested some time ago by Eckhaus (1965); as we shall see below, it gives results consistent with those obtained by standard bifurcation methods, but has greater scope.

In the governing equations (2.1)–(2.3), it is convenient to set

$$\rho = R^{\frac{1}{2}}, \tag{4.1}$$

and to use ρ as the substantive parameter rather than R. Let $(\mathbf{v}_{mjn}, \theta_{mjn})$ denote an eigensolution of the form (3.7) of the linear stability problem, with ρ_{mjn} the corresponding eigenvalue. Let $(\mathbf{v}, \theta)$ be a vector that satisfies (2.2) and the boundary conditions (2.4)–(2.5). For any number ρ consider the expression

$$Q \equiv \langle \mathbf{v}_{mjn} \cdot (\nabla^2 \mathbf{v} - \nabla p + \rho\theta\hat{\mathbf{z}}) + \theta_{mjn}(\nabla^2\theta + \rho w)\rangle, \tag{4.2}$$

where $\langle\ \rangle$ denotes integration over the volume occupied by the fluid. From an integration by parts and the fact that $(\mathbf{v}_{mjn}, \theta_{mjn})$ solves the linear stability problem it is easy to show that

$$Q = (\rho - \rho_{mjn})\langle w_{mjn}\theta + \theta_{mjn} w\rangle; \tag{4.3}$$

hence $Q = 0$ when $\rho = \rho_{mjn}$.

Now let $(\mathbf{v}, \theta)$ denote a solution of the system (2.1)–(2.3) with boundary conditions (2.4)–(2.5) for some given value of ρ. Take the scalar product of (2.1) with $\mathbf{v}_{mjn}$ and of (2.3) with θ_{mjn}, add, and integrate over the fluid volume. Using (4.3) we find that the outcome of these operations is the equation

$$(\rho - \rho_{mjn})\langle w_{mjn}\theta + \theta_{mjn} w\rangle = \left\langle \theta_{mjn}\frac{\partial\theta}{\partial t} + Pr^{-1}\,\mathbf{v}_{mjn}\cdot\frac{\partial \mathbf{v}}{\partial t}\right\rangle + \langle \theta_{mjn}(\mathbf{v}\,.\,\nabla)\,\theta + Pr^{-1}\,\mathbf{v}_{mjn}(\mathbf{v}\,.\,\nabla)\,\mathbf{v}\rangle. \tag{4.4}$$

Next we choose a finite set S of eigensolutions $(\mathbf{v}_{mjn}, \theta_{mjn})$ of the linear stability problem. For convenience we write

$$S = \{mjn\}, \tag{4.5}$$

which signifies that an element of the set has azimuthal, radial and vertical wavenumbers m, j and n respectively; we can thus refer to $mjn = p$, say, as the *vector wavenumber* of an element of S. Also, let $N \geqslant 1$ be the number of elements in the set S.

These elements have important orthogonality properties: let $(\mathbf{v}_p, \theta_p)$, $(\mathbf{v}_q, \theta_q)$ denote two elements with vector wavenumbers p, q respectively. Then it is easily shown from (3.7) that

$$\langle \theta_p \theta_q \rangle = \langle \mathbf{v}_p . \mathbf{v}_q \rangle = \langle \theta_p w_q \rangle = 0 \quad \text{when} \quad p \neq q. \tag{4.6}$$

Our fundamental assumption is that the solution vector $(\mathbf{v}, \theta)$ can be represented, to a good approximation, by a linear combination of elements of the finite set S, with time-dependent coefficients. We postpone for the present the question of the validity of this assumption, and set

$$(\mathbf{v}, \theta) = \sum_s A_{mjn}(t)\,(\mathbf{v}_{mjn}, \theta_{mjn}) = \sum_s A_p(t)\,(\mathbf{v}_p, \theta_p), \tag{4.7}$$

where p denotes the vector wavenumber, as before.

We substitute (4.7) into (4.4); using the orthogonality relations (4.6), we obtain a system of equations of the form

$$\nu_p \frac{dA_p}{dt} = (\rho - \rho_p)\,A_p - Z_p \quad (p = 1, \ldots, N), \tag{4.8}$$

where N is the number of elements in S. In (4.8) it is easy to show that the coefficients ν_p are given by

$$2\langle \theta_p w_p \rangle\, \nu_p = \langle \theta_p^2 + Pr^{-1}\, \mathbf{v}_p . \mathbf{v}_p \rangle, \tag{4.9}$$

and that the Z_p are homogeneous quadratic functions of the A_p, defined by

$$2\langle \theta_p w_p \rangle\, Z_p = \langle \theta_p (\sum_s A_q \mathbf{v}_q . \nabla) \sum_s A_r \theta_r + Pr^{-1}\, \mathbf{v}_p . (\sum_s A_q \mathbf{v}_q . \nabla) \sum_s A_r \mathbf{v}_r \rangle. \tag{4.10}$$

We note from (3.7) that

$$d_p \equiv 2\langle \theta_p w_p \rangle = 2C_p \langle \theta_p^2 \rangle, \tag{4.11}$$

while from (3.8) we see that $C_p > 0$ for every eigensolution. Hence we deduce the inequalities

$$d_p > 0, \quad \nu_p > 0, \tag{4.12}$$

for every element of S; these will be useful later.

We consider now the question of how to decide on a suitable set S. To study the onset of convection as ρ increases through the critical value ρ_c (for a fixed aspect ratio), we take S to comprise the critical mode and the minimum number of other modes required to generate nonlinear evolution into convection of the critical mode. We shall call this S the minimal set for the onset of convection, noting, however, that, as the critical mode changes with aspect ratio, so will the corresponding minimal set. For values of ρ not much greater than ρ_c it is permissible to neglect modes of wavenumbers p when $\rho_p \gg \rho$. This is justified by the fact that, because of the form of (4.8), modes having $\rho_p \gg \rho$ are strongly damped except when generated by the quadratic self-interaction of the critical mode. A general discussion of the issues involved in the choice of S can be found in Rosenblat (1979).

It is worth recalling that the minimal set cannot comprise the critical mode alone. From (3.7) it is easily shown that

$$\langle \theta_p \mathbf{v}_p . \nabla \theta_p \rangle = \langle \mathbf{v}_p . \mathbf{v}_p . \nabla \mathbf{v}_p \rangle = 0, \tag{4.13}$$

for any p. Hence, if S were to consist of just one mode, (4.10) would give $Z_p = 0$, and then (4.8) would not reveal any nonlinear evolution of the mode.

We use these ideas to investigate the behaviour of the system at various values of the aspect ratio.

5. Bifurcation at simple points

In this section we study weakly nonlinear evolution into convection at three specific values of the aspect ratio, namely $a = 0{\cdot}8$, $1{\cdot}4$ and $1{\cdot}7$. As can be seen from figure 1, the loss of stability of the conduction state at each of these values is *simple*, that is, only one mode loses stability as the Rayleigh number increases through its critical value. In each case we reduce the nonlinear problem to a Landau evolution equation, and show that at each point there is supercritical bifurcation into stable convection. This is in accordance with well-known results for the horizontally unbounded layer (see e.g. Palm 1975).

5.1. *Bifurcation when $a = 0{\cdot}8$*

Figure 1 shows that the critical mode at this value is 111, that is, azimuthal, radial and vertical wavenumbers are all equal to one. From (3.5) we find that

$$R_c = R_{111} = 659. \tag{5.1}$$

To determine the minimal set S we observe that the quadratic self-interaction of the mode 111 generates the modes $0j2$, $2j2$, with $j = 1, 2, 3, \ldots$. One can easily show, however, that

$$R_{m12} < R_{m22} < R_{m32} < \ldots \quad (m = 0, 2). \tag{5.2}$$

For this reason we neglect all but the leading modes (in the sense of the ordering (5.2)) and take the minimal set to be

$$S = \{111, 012, 212\}. \tag{5.3}$$

It can in fact be shown that the modes $0j2$, $2j2$ for $j \geqslant 2$ have no qualitative effect on the behaviour of the system.

We need to consider (4.8) with the quadratic nonlinearities given by (4.10). For the mode 111, the first term on the right-hand side of (4.10) is

$$\langle \theta_{111}(A_{111}\mathbf{v}_{111} + A_{012}\mathbf{v}_{012} + A_{212}\mathbf{v}_{212}) . \nabla(A_{111}\theta_{111} + A_{012}\theta_{012} + A_{212}\theta_{212})\rangle, \tag{5.4}$$

which contains altogether nine integrals over the fluid volume. On substituting from (3.7), however, we show easily that, because of the periodicity of the eigenvectors in the ϕ- and z-directions, seven of these nine integrals are identically zero. In a similar fashion, the second term on the right of (4.10) has nine integrals of products of eigenvectors, but again only two of these do not vanish identically. Eventually, then, we can show that the quadratic nonlinearity associated with the mode 111 is given by

$$d_{111} Z_{111} = A_{111}(\alpha_0 A_{012} + \alpha_2 A_{212}), \tag{5.5}$$

where d_{111} is the inner product defined by (4.11), and where

$$\alpha_m = \langle \theta_{111}\mathbf{v}_{111} . \nabla\theta_{m12} + Pr^{-1}\mathbf{v}_{111} . (\mathbf{v}_{111} . \nabla)\mathbf{v}_{m12}\rangle \quad (m = 0, 2). \tag{5.6}$$

A reduction of the quadratic nonlinearities Z_{012} and Z_{212} can be achieved using similar arguments. After performing some elementary algebraic manipulations we find that

$$d_{m12} Z_{m12} = -\alpha_m A_{111}^2 \quad (m = 0, 2), \tag{5.7}$$

where the α_m are given by (5.6).

Equations (4.8) for the set (5.3) now take the following forms:

$$\nu_{111} A'_{111} = (\rho - \rho_c) A_{111} - (\alpha_0 A_{012} + \alpha_2 A_{212}) A_{111}/d_{111}, \tag{5.8}$$

$$\nu_{012} A'_{012} = (\rho - \rho_{012}) A_{012} + \alpha_0 A^2_{111}/d_{012}, \tag{5.9}$$

$$\nu_{212} A'_{212} = (\rho - \rho_{212}) A_{212} + \alpha_2 A^2_{111}/d_{212}, \tag{5.10}$$

where the prime denotes differentiation with respect to t, and where we have put $\rho_{111} = \rho_c$, the critical value. The system (5.8)–(5.10) constitutes the evolution equations for the mode 111 at aspect ratio $a = 0{\cdot}8$. The null solution $A_{111} = A_{012} = A_{212} = 0$ corresponds to the conduction state, and is stable for $\rho < \rho_c$ and unstable for $\rho > \rho_c$.

The system (5.8)–(5.10) can be simplified for weakly nonlinear evolution in the following way. We find from (3.5) that

$$\rho_{111} \approx 25{\cdot}7, \quad \rho_{012} \approx 103{\cdot}0, \quad \rho_{212} \approx 104{\cdot}1, \tag{5.11}$$

when $a = 0{\cdot}8$. Consequently, in the neighbourhood of $\rho = \rho_{111} = \rho_c$ any initial disturbances corresponding to the modes 012 and 212 are strongly damped, and their presence is due to the quadratic self-interaction of the critical mode 111. Hence we can neglect the time-derivative terms in (5.9) and (5.10), and also replace ρ by ρ_c to a good approximation, to obtain

$$A_{012} = \frac{\alpha_0 A^2_{111}}{d_{012}(\rho_{012} - \rho_c)}, \quad A_{212} = \frac{\alpha_2 A^2_{111}}{d_{212}(\rho_{212} - \rho_c)}. \tag{5.12}$$

Substituting these into (5.8), we obtain

$$\nu_{111} A'_{111} = (\rho - \rho_c) A_{111} - \omega_{111} A^3_{111}, \tag{5.13}$$

where

$$\omega_{111} = \frac{\alpha_0^2}{d_{111} d_{012}(\rho_{012} - \rho_c)} + \frac{\alpha_2^2}{d_{111} d_{212}(\rho_{212} - \rho_c)}. \tag{5.14}$$

Equation (5.13) is the Landau equation for the evolution of the critical mode 111 (the fact that the linear growth rate appears as $\rho - \rho_c$ instead of the usual $R - R_c$ is a technicality). Because of the inequalities (4.12) we see that

$$\omega_{111} > 0, \tag{5.15}$$

without needing to compute any coefficients. The form of (5.13) is thus standard: a solution bifurcates from the critical value ρ_c having the representation

$$A_{111} = \pm [(\rho - \rho_c)/\omega_{111}]^{\frac{1}{2}} \tag{5.16}$$

for its amplitude, and having the form of a non-axisymmetric mode with azimuthal wavenumber 1. This solution exists only for $\rho > \rho_c$ (supercritically) and is known from elementary bifurcation theory to be stable.

5.2. *Bifurcation when $a = 1{\cdot}4$*

The critical mode at this aspect ratio is 211: the azimuthal wavenumber is 2, and the radial and vertical wavenumbers are both 1. The critical Rayleigh number is

$$R_c = R_{211} = 658. \tag{5.17}$$

The quadratic self-interaction of the mode 211 generates the modes $0j2$, $4j2$ with

$j = 1, 2, 3, \ldots$. By the same reasoning as before we neglect the modes with $j \geqslant 2$, and take the minimal set to be

$$S = \{211, 012, 412\}. \tag{5.18}$$

After performing the standard calculations, we find for the nonlinearity associated with the critical mode 211

$$d_{211} Z_{211} = A_{211}(\beta_0 A_{012} + \beta_4 A_{412}), \tag{5.19}$$

where

$$\beta_m = \langle \theta_{211} \mathbf{v}_{211} \cdot \nabla \theta_{m12} + Pr^{-1} \mathbf{v}_{211} \cdot (\mathbf{v}_{211} \cdot \nabla) \mathbf{v}_{m12} \rangle \quad (m = 0, 4), \tag{5.20}$$

while the nonlinearities associated with the modes 012 and 412 are found to be given by

$$d_{m12} Z_{m12} = -\beta_m A_{211}^2 \quad (m = 0, 4). \tag{5.21}$$

The three evolution equations (4.8) for the set (5.17) can now be written down. As before, however, we reduce them to a single equation by setting $A'_{012} = A'_{412} = 0$, and solving for A_{012}, A_{412} in terms of A_{211}. We obtain in this way the Landau equation

$$\nu_{211} A'_{211} = (\rho - \rho_c) A_{211} - \omega_{211} A_{211}^3, \tag{5.22}$$

where

$$\omega_{211} = \frac{\beta_0^2}{d_{211} d_{012}(\rho_{012} - \rho_c)} + \frac{\beta_4^2}{d_{211} d_{412}(\rho_{412} - \rho_c)}, \tag{5.23}$$

for ρ sufficiently close to ρ_{211}. As before we see that $\omega_{211} > 0$, and so the expression

$$A_{211} = \pm [(\rho - \rho_c)/\omega_{211}]^{\frac{1}{2}} \tag{5.24}$$

represents the amplitude of a stable, supercritical convection solution having azimuthal wavenumber 2 and bifurcating from the critical point $\rho = \rho_c$.

5.3. *Bifurcation when* $a = 1{\cdot}7$

At this value of the aspect ratio the critical mode is the axisymmetric mode 011, whose quadratic self-interaction generates the modes $0j2$, with $j = 1, 2, 3, \ldots$. We truncate as before, and take the minimal set to be

$$S = \{011, 012\}. \tag{5.25}$$

We evaluate the nonlinear quantities (4.10) to obtain

$$d_{011} Z_{011} = \gamma_{00} A_{011} A_{012}, \quad d_{012} Z_{012} = -\gamma_{00} A_{011}^2, \tag{5.26}$$

where

$$\gamma_{00} = \langle \theta_{011} \mathbf{v}_{011} \cdot \nabla \theta_{012} + Pr^{-1} \mathbf{v}_{011} \cdot (\mathbf{v}_{011} \cdot \nabla) \mathbf{v}_{012} \rangle. \tag{5.27}$$

Writing down the evolution equations and solving for A_{012}, we arrive finally at the Landau equation

$$\nu_{011} A'_{011} = (\rho - \rho_c) A_{011} - \omega_{011} A_{011}^3, \tag{5.28}$$

where

$$\omega_{011} = \frac{\gamma_{00}^2}{d_{011} d_{012}(\rho_{012} - \rho_c)} > 0. \tag{5.29}$$

From (5.28)–(5.29) we infer that there is supercritical bifurcation into stable axisymmetric convection at the aspect ratio $a = 1{\cdot}7$.

6. Bifurcation near double points

It is evident from figure 1 that there are certain values of the aspect ratio at which two modes lose stability simultaneously. At these values the Rayleigh number is a double eigenvalue of the linear stability problem. In this section we shall investigate the onset of convection in the neighbourhood of such double points.

6.1. *Bifurcation near point A*

At the point marked A on figure 1 the curves of R as a function of aspect ratio for the modes 111 and 211 intersect. We denote by a_A the value of the aspect ratio at which this occurs; in fact $a_A \approx 1{\cdot}08$. The common value of R_{111}, R_{211} at this point will be denoted R_A, and ρ_A will be the corresponding value of ρ.

We are interested in studying the onset of convection at values of a close to a_A. As noted in § 5, the self-interaction of the mode 111 generates modes $0j2$, $2j2$ ($j = 1, 2, \ldots$), while the self-interaction of the mode 211 generates modes $0j2$, $4j2$. In addition, the mutual interaction of the modes 111, 211 generates $1j2$, $3j2$. To investigate the behaviour of the system in a neighbourhood of $a = a_A$ we neglect the modes with $j \geqslant 2$ and take as the minimal set

$$S = \{111, 211, 012, 112, 212, 312, 412\}. \tag{6.1}$$

This includes the two critical modes and their dominant interactions.

The system (4.8) now comprises seven evolution equations, which are conveniently grouped in the following way:

$$\nu_{m11} A'_{m11} = (\rho - \rho_{m11}) A_{m11} - Z_{m11} \quad (m = 1, 2), \tag{6.2}$$

$$0 = (\rho_A - \rho_{m12}) A_{m12} - Z_{m12} \quad (m = 0, \ldots, 4), \tag{6.3}$$

where the derivatives in the second group of equations have been set equal to zero, in accordance with the discussion in § 5.

The quadratic nonlinearities appearing in (6.2) are calculated from (4.10). We find that

$$d_{111} Z_{111} = A_{111}(\alpha_0 A_{012} + \alpha_2 A_{212}) + A_{211}(\alpha_{21} A_{112} + \alpha_{23} A_{312}), \tag{6.4}$$

where α_0, α_2 are defined by (5.6), and where

$$\alpha_{2m} = \langle \theta_{111}(\mathbf{v}_{211} . \nabla\theta_{m12} + \mathbf{v}_{m12} . \nabla\theta_{211}) + Pr^{-1}\, \mathbf{v}_{111} . (\mathbf{v}_{211} . \nabla\mathbf{v}_{m12} + \mathbf{v}_{m12} . \nabla\mathbf{v}_{211})\rangle \tag{6.5}$$

for $m = 1, 3$; and

$$d_{211} Z_{211} = A_{211}(\beta_0 A_{012} + \beta_4 A_{412}) + A_{111}(\beta_{21} A_{112} + \beta_{23} A_{312}), \tag{6.6}$$

where β_0, β_4 are defined by (5.20) and where

$$\beta_{2m} = \langle \theta_{211}(\mathbf{v}_{111} . \nabla\theta_{m12} + \mathbf{v}_{m12} . \nabla\theta_{111}) + Pr^{-1}\, \mathbf{v}_{211} . (\mathbf{v}_{111} . \nabla\mathbf{v}_{m12} + \mathbf{v}_{m12} . \nabla\mathbf{v}_{111})\rangle \tag{6.7}$$

for $m = 1, 3$. The quadratic nonlinearities in (6.3) are also determined from (4.10), and are as follows:

$$d_{012} Z_{012} = -\alpha_0 A_{111}^2 - \beta_0 A_{211}^2, \tag{6.8a}$$

$$d_{212} Z_{212} = -\alpha_2 A_{111}^2, \quad d_{412} Z_{412} = -\beta_4 A_{211}^2, \tag{6.8b}$$

$$d_{112} Z_{112} = -\delta_1 A_{111} A_{211}, \quad d_{312} Z_{312} = -\delta_3 A_{111} A_{211}, \tag{6.8c}$$

Pr	ν_{111}	$\omega_{111}\times 10^3$	$\sigma_{111}\times 10^2$	ν_{211}	$\sigma_{211}\times 10^2$	$\omega_{211}\times 10^3$
1·0	2·09	0·78	0·64	1·49	0·82	1·45
10	1·15	0·26	0·52	0·82	0·30	0·53
∞	1·06	0·23	0·51	0·75	0·26	0·47

TABLE 1

where

$$\delta_m = \langle(\theta_{211}\mathbf{v}_{111}.\nabla+\theta_{111}\mathbf{v}_{211}.\nabla)\,\theta_{m12}+Pr^{-1}[\mathbf{v}_{211}.(\mathbf{v}_{111}.\nabla)+\mathbf{v}_{111}.(\mathbf{v}_{211}.\nabla)]\,\mathbf{v}_{m12}\rangle \tag{6.9}$$

for $m = 1, 3$.

Substituting (6.3) and (6.8) into (6.4) and (6.6), we can rewrite (6.2) as a pair of coupled evolution equations for the modes A_{111} and A_{211}:

$$\nu_{111}A'_{111} = (\rho-\rho_{111})\,A_{111}-\omega_{111}A^3_{111}-\sigma_{111}A_{111}A^2_{211}, \tag{6.10a}$$

$$\nu_{211}A'_{211} = (\rho-\rho_{211})\,A_{211}-\sigma_{211}A^2_{111}A_{211}-\omega_{211}A^3_{211}. \tag{6.10b}$$

In these equations ω_{111}, ω_{211} are as defined by (5.14) and (5.23) respectively, with ρ_c replaced by ρ_A, and the coefficients $\sigma_{111}, \sigma_{211}$ are given by

$$d_{111}\sigma_{111} = \frac{\alpha_0\beta_0}{d_{012}(\rho_{012}-\rho_A)}+\frac{\alpha_{21}\delta_1}{d_{112}(\rho_{112}-\rho_A)}+\frac{\alpha_{23}\delta_3}{d_{312}(\rho_{312}-\rho_A)}, \tag{6.11a}$$

$$d_{211}\sigma_{211} = \frac{\alpha_0\beta_0}{d_{012}(\rho_{012}-\rho_A)}+\frac{\beta_{21}\delta_1}{d_{112}(\rho_{112}-\rho_A)}+\frac{\beta_{23}\delta_3}{d_{312}(\rho_{312}-\rho_A)}. \tag{6.11b}$$

Computed values of the coefficients in (6.10a, b) for various values of the Prandtl number are shown in table 1.

We consider first the nature and stability of solutions at aspect ratios slightly less than a_A. Then we introduce the notation

$$\eta = \rho-\rho_{111}, \quad \epsilon = \rho_{211}-\rho_{111} > 0, \tag{6.12}$$

and it is convenient to write (6.10a, b) in the form

$$\nu_1A'_1 = \eta A_1-\omega_1A^3_1-\sigma_1A_1A^2_2, \tag{6.13a}$$

$$\nu_2A'_2 = (\eta-\epsilon)\,A_2-\sigma_2A^2_1A_2-\omega_2A^3_2, \tag{6.13b}$$

where for the sake of brevity we have replaced the subscripts 111, 211 by 1, 2 respectively.

We note that, if $(\bar{A}_1, \bar{A}_2)$ denotes a solution of (6.13), the stability of this solution to a disturbance (a_1, a_2) is determined from the exponents of the linear system

$$\nu_1a'_1 = (\eta-3\omega_1\bar{A}^2_1-\sigma_1\bar{A}^2_2)\,a_1-2\sigma_1\bar{A}_1\bar{A}_2a_2, \tag{6.14a}$$

$$\nu_2a'_2 = -2\sigma_2\bar{A}_1\bar{A}_2a_1+(\eta-\epsilon-\sigma_2\bar{A}^2_1-3\omega_2\bar{A}^2_2)\,a_2. \tag{6.14b}$$

The system (6.13) has effectively four solutions.

I. The trivial solution (the conduction state) $A_1 = A_2 = 0$ exists for all η, and is stable for $\eta < 0$ and unstable for $\eta > 0$; in other words, this solution exists for all ρ, and is stable for $\rho < \rho_{111}$ and unstable for $\rho > \rho_{111}$.

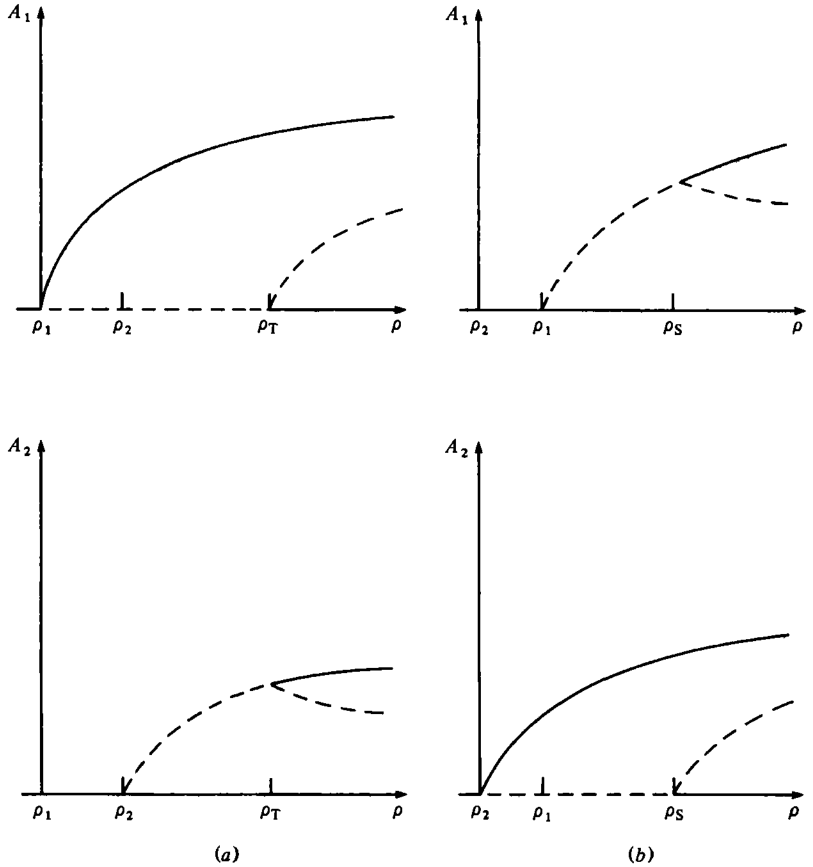

FIGURE 2. Bifurcation diagrams for solutions: (*a*) just to the left of point *A* in figure 1; and (*b*) just to the right of point *A*. ——, stable solutions; – – –, unstable solutions.

II. There is a solution (actually a pair of solutions)

$$A_1^2 = \bar{A}_1^2 = \eta/\omega_1, \quad A_2^2 = 0 \tag{6.15}$$

for $\eta > 0$ ($\rho > \rho_{111}$); this is the standard supercritical bifurcating solution. For this solution the stability equations reduce to

$$\nu_1 a_1' = -2\eta a_1, \quad \nu_2 a_2' = (\eta - \epsilon - \sigma_2 \eta/\omega_1)\, a_2. \tag{6.16}$$

Since $\sigma_2/\omega_1 > 1$ (table 1), it follows that this solution is always *stable*.

III. The solution

$$A_1^2 = 0, \quad A_2^2 = \bar{A}_2^2 = (\eta - \epsilon)/\omega_2 \tag{6.17}$$

bifurcates to the right from the second critical point ρ_{211}. For its stability we have

$$\nu_1 a_1' = [\eta - \sigma_1(\eta - \epsilon)/\omega_2]\, a_1, \quad \nu_2 a_2' = -2(\eta - \epsilon)\, a_2, \tag{6.18}$$

from which we infer that this solution is unstable initially, when η is slightly greater than ϵ, but regains stability at the point

$$\eta = \eta_T = \frac{\sigma_1 \epsilon}{\sigma_1 - \omega_2} \quad \text{or} \quad \rho = \rho_T = \rho_{211} + \frac{\sigma_1 \epsilon}{\sigma_1 - \omega_2}. \tag{6.19}$$

IV. There is a mixed solution

$$A_1^2 = \bar{A}_1^2 = \frac{(\eta-\epsilon)\,\sigma_1-\eta\omega_2}{\sigma_1\sigma_2-\omega_1\omega_2}, \quad A_2^2 = \bar{A}_2^2 = \frac{\eta\sigma_2-(\eta-\epsilon)\,\omega_1}{\sigma_1\sigma_2-\omega_1\omega_2}, \tag{6.20}$$

which is easily seen to exist only for $\eta > \eta_T$. In fact, this solution bifurcates from the solution III at the point where the latter changes from unstable to stable. A simple calculation involving (6.14*a, b*) shows that the solution (6.20) is *unstable*.

A schematic representation of all these solutions is shown in figure 2(*a*); stable solutions are indicated by solid lines and unstable solutions by dotted lines. The conduction solution I is stable for $\rho < \rho_{111}$, and the pure-mode (with azimuthal wave-number $m = 1$) convection solution II is stable for $\rho > \rho_{111}$. In addition, there is another stable pure mode (with $m = 2$) for $\rho > \rho_T$. Numerical calculations show that ρ_T increases with decreasing Prandtl number. The secondary bifurcation into a mixed mode is always unstable.

For aspect ratios slightly greater than a_A, we have that $\rho_{211} < \rho_{111}$ and the foregoing analysis can be repeated if instead of (6.12) we use the definitions

$$\eta = \rho-\rho_{211}, \quad \epsilon = \rho_{111}-\rho_{211}. \tag{6.21}$$

It is then easy to show that the pure-mode ($m = 2$) convection solution, which bifurcates from the conduction solution at $\rho = \rho_{211}$, exists and is stable for $\rho > \rho_{211}$. The convection solution with $m = 1$ originates at $\rho = \rho_{111}$, is initially unstable and regains stability at ρ_S given by

$$\rho_S = \rho_{111}+\frac{\sigma_2\epsilon}{\sigma_2-\omega_1}. \tag{6.22}$$

There is also secondary bifurcation into an unstable mixed mode. The pattern of solutions is illustrated in figure 2(*b*).

6.2. *Bifurcation near point B*

The point B in figure 1 is the intersection of the curves $R = R(a)$ for the modes 211 and 011. At this point we write $a = a_B$ ($\approx$ 1·54) and $R_{211} = R_{011} = R_B$. For convection in a neighbourhood of a_B we take

$$S = \{211, 011, 012, 212, 412\}; \tag{6.23}$$

this is the minimal set describing the self-interactions and mutual interaction of the modes 211, 011.

The system (4.8) is now composed of five equations:

$$\nu_{m11}A'_{m11} = (\rho-\rho_{m11})\,A_{m11}-Z_{m11} \quad (m = 0, 2), \tag{6.24}$$

$$A_{m12} = \frac{Z_{m12}}{\rho_B-\rho_{m12}} \quad (m = 0, 2, 4). \tag{6.25}$$

The quadratic nonlinearities are calculated from (4.10) in the usual way. We obtain

$$d_{211}Z_{211} = A_{211}(\beta_0 A_{012}+\beta_4 A_{412})+\gamma_{02}A_{011}A_{212}, \tag{6.26}$$

where β_0, β_4 are as previously defined and where

$$\gamma_{02} = \langle\theta_{211}(\mathbf{v}_{011}\cdot\nabla\theta_{212}+\mathbf{v}_{212}\cdot\nabla\theta_{011})+Pr^{-1}\,\mathbf{v}_{211}\cdot(\mathbf{v}_{011}\cdot\nabla\mathbf{v}_{212}+\mathbf{v}_{212}\cdot\nabla\mathbf{v}_{011})\rangle; \tag{6.27}$$

Pr	ν_{211}	$\omega_{211}\times 10^3$	$\tau_{211}\times 10^2$	ν_{011}	$\tau_{011}\times 10^4$	$\omega_{011}\times 10^2$
1·0	1·86	1·23	0·29	1·61	1·06	0·75
10	1·03	0·47	0·23	0·89	0·43	0·33
∞	0·95	0·41	0·22	0·81	0·39	0·30

TABLE 2

and

$$d_{011}Z_{011} = \gamma_{00}A_{011}A_{012}+\gamma_{22}A_{211}A_{212}, \tag{6.28}$$

where

$$\gamma_{22} = \langle\theta_{011}(\mathbf{v}_{211}.\nabla\theta_{212}+\mathbf{v}_{212}.\nabla\theta_{211})+Pr^{-1}\,\mathbf{v}_{011}.(\mathbf{v}_{211}.\nabla\mathbf{v}_{212}+\mathbf{v}_{212}.\nabla\mathbf{v}_{211})\rangle. \tag{6.29}$$

We also find that

$$d_{012}Z_{012} = -\gamma_{00}A_{011}^2-\beta_0A_{211}^2, \tag{6.30a}$$

$$d_{212}Z_{212} = -\gamma_2A_{211}A_{011},\quad d_{412}Z_{412} = -\beta_4A_{211}^2, \tag{6.30b, c}$$

where

$$\gamma_2 = \langle(\theta_{011}\mathbf{v}_{211}.\nabla+\theta_{211}\mathbf{v}_{011}.\nabla)\,\theta_{212}+Pr^{-1}[\mathbf{v}_{011}.(\mathbf{v}_{211}.\nabla)+\mathbf{v}_{211}.(\mathbf{v}_{011}.\nabla)]\,\mathbf{v}_{212}\rangle. \tag{6.31}$$

Some simple algebra enables us to reduce (6.24) to a pair of coupled evolution equations for the modes A_{211} and A_{011}:

$$\nu_{211}A_{211}' = (\rho-\rho_{211})A_{211}-\omega_{211}A_{211}^3-\tau_{211}A_{211}A_{011}^2, \tag{6.32a}$$

$$\nu_{011}A_{011}' = (\rho-\rho_{011})A_{011}-\tau_{011}A_{211}^2A_{011}-\omega_{011}A_{011}^3, \tag{6.32b}$$

where $\omega_{211}, \omega_{011}$ are defined by (5.23), (5.29) respectively, and where

$$\tau_{211} = \frac{\beta_0\gamma_{00}}{d_{211}d_{012}(\rho_{012}-\rho_B)}+\frac{\gamma_2\gamma_{02}}{d_{211}d_{212}(\rho_{212}-\rho_B)}, \tag{6.33a}$$

$$\tau_{011} = \frac{\beta_0\gamma_{00}}{d_{011}d_{012}(\rho_{012}-\rho_B)}+\frac{\gamma_2\gamma_{22}}{d_{011}d_{212}(\rho_{212}-\rho_B)}. \tag{6.33b}$$

Computed values of the respective coefficients are shown in table 2.

We proceed to analyse the system (6.32) as in the previous case. When a is slightly less than a_B, we write

$$\eta = \rho-\rho_{211},\quad \epsilon = \rho_{011}-\rho_{211} > 0, \tag{6.34}$$

whereupon (6.32a, b) become, in abbreviated notation,

$$\nu_2A_2' = \eta A_2-\omega_2A_2^3-\tau_2A_1A_2^2, \tag{6.35a}$$

$$\nu_0A_0' = (\eta-\epsilon)A_0-\tau_0A_2^2A_0-\omega_0A_0^3. \tag{6.35b}$$

These equations are identical in structure with (6.13), and the analysis is performed in the same way. There are, however, important differences in behaviour due to differences in the numerical values of the coefficients. We omit the computational details and merely report the results, as follows.

The conduction solution loses stability at $\eta = 0$ ($\rho = \rho_{211}$), and is replaced by a supercritically bifurcating solution that has azimuthal wavenumber 2. This solution

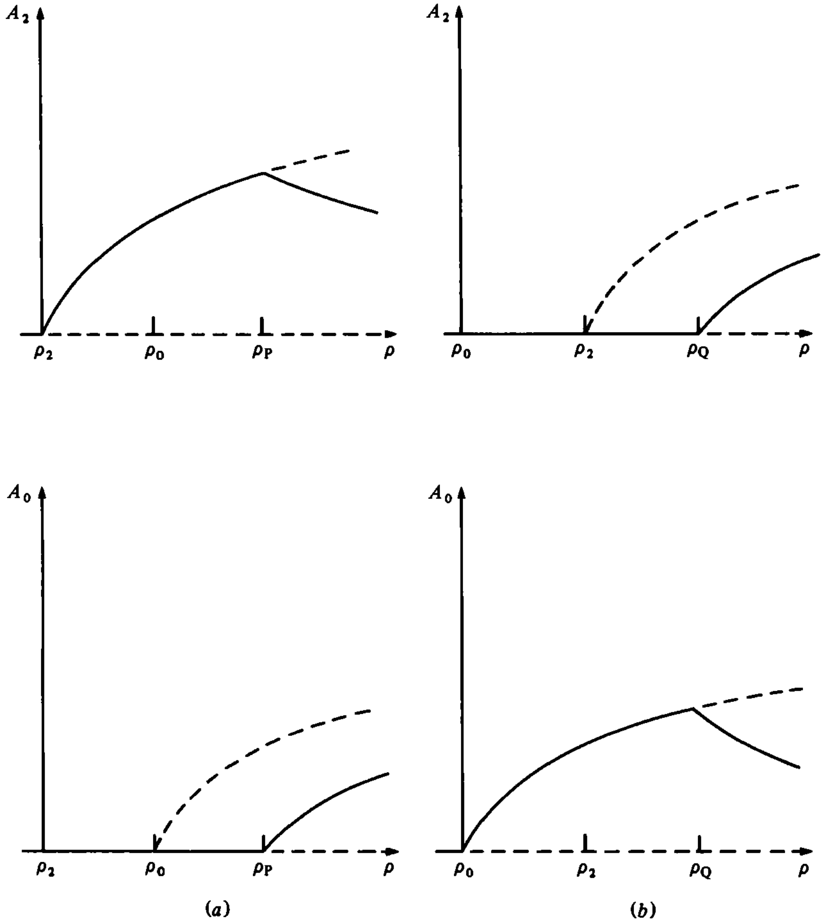

FIGURE 3. Bifurcation diagrams for solutions: (*a*) just to the left of point *B* in figure 1; and (*b*) just to the right of point *B*. ——, stable solutions; – – –, unstable solutions.

is stable when it leaves the bifurcation point but, since $\tau_0/\omega_2 < 1$, it becomes *unstable* at the point

$$\eta = \eta_P = \frac{\omega_2 \epsilon}{\omega_2 - \tau_0}. \tag{6.36}$$

The axisymmetric convective solution emerging supercritically from $\eta = \epsilon$ is always unstable. The mixed solution

$$A_2^2 = \frac{\eta\omega_0 - (\eta - \epsilon)\tau_2}{\omega_0\omega_2 - \tau_1\tau_2}, \quad A_0^2 = \frac{(\eta-\epsilon)\,\omega_2 - \eta\tau_0}{\omega_0\omega_2 - \tau_0\tau_2} \tag{6.37}$$

exists only for $\eta > \eta_P$; this solution appears as a secondary bifurcation from the $m = 2$ solution at the point where the latter changes from stable to unstable. The solution (6.37) can be shown to be *stable*. All these solutions are illustrated in figure 3 (*a*).

When a is slightly greater than a_B, we repeat the analysis with

$$\eta = \rho - \rho_{011}, \quad \epsilon = \rho_{211} - \rho_{011} > 0, \tag{6.38}$$

and we find the same pattern of behaviour, with the roles of the modes reversed. As shown in figure 3(*b*), the stable solutions are the conduction solution for $\eta < 0$, an axisymmetric convection solution for $0 < \eta < \eta_Q$, where

$$\eta_Q = \frac{\epsilon\omega_0}{\omega_0 - \tau_2}, \tag{6.39}$$

and a mixed-mode solution for $\eta > \eta_Q$.

7. Discussion

When the critical Rayleigh number is a simple eigenvalue of the linear stability problem, the convective motion has the geometry of the critical mode. (This, of course, is a classical result of bifurcation theory.) In particular, when $a < 1{\cdot}0$ approximately the incipient convection is a mode of azimuthal wavenumber 1 and, since the two lowest Rayleigh number curves are well separated in this regime, one would expect this pure mode to persist over quite a range of supercritical Rayleigh numbers.

When $a > 1{\cdot}0$, on the other hand, the neutral stability curves begin to bunch together, and the typical behaviour is likely to be that encountered in the neighbourhoods of double points of criticality, as discussed in §6. One may infer from that discussion that either one of two pure modes can exist when a is slightly greater than one, but that with further increase in aspect ratio a mixed mode is the predominant form of the convection.

The latter conclusion is broadly consistent with that of Charlson & Sani (1975), who studied the stability of axisymmetric convection to non-axisymmetric disturbances under a variety of boundary conditions. They found in many cases that axisymmetric convection became unstable at some supercritical Rayleigh number; presumably, according to our calculations, it would be replaced by mixed-mode convection. The results of Rosenblat *et al.* (1982) on Marangoni convection bear some resemblance to those obtained above in that pure modes are generally replaced by mixed modes near double points, but there are also significant differences due to the presence of quadratic nonlinearities in the evolution equations of the Marangoni problem.

As the aspect ratio increases the neutral stability curves become very close, and a separate analysis is required. This will be presented in a later paper.

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